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LETTER TO THE EDITOR

The two-element spin chain related to the Heisenberg spin model

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Abstract. We construct a two-element spin-chain model based on the periodic Heisenberg spin model. This model describes a system containing two kinds of particle. The Bethe *ansatz* equations are obtained by using the quantum inverse scattering method. The high- and low-temperature limits are discussed, and we find that the mean energy per site increases with increase of the next-nearest-neighbour coupling in the high-temperature limit.

Introduction

The exact solution of the one-dimensional Heisenberg model [1] has been treated by many authors and has been evoked to some extent in using the Yang–Baxter equation. This equation was firstly discovered by Yang and Yang [2]. It appeared in the problem of non-relativistic (1 + 1)-dimensional particles with δ-function interaction, as the self-consistency condition for Bethe's *ansatz*. An analogous relation was derived by Baxter [3], who investigated the eight-vertex lattice model which can be used to discuss the anisotropic Heisenberg spin chain. These relations guarantee the commutativity of transfer matrices with different values of the anisotropy parameters λ. With the development of nanofabrication techniques for quantum wires and the prediction of edge states in the quantum Hall effect, the interest in one-dimensional electron systems has been renewed in recent years [4–8], and the exact expressions for the surface energies, the low-temperature specific heats, the Pauli susceptibilities, and the Kondo temperatures of the systems have been established. Much progress has been made recently as regards impurity models based on the methods of renormalization-group techniques [9], conformal field theory [10], and integrability investigations. The quantum inverse scattering method [11] and the Bethe *ansatz* technique [12] have been found to be powerful tools for studying integrable models within the framework of quantum spin chains [13, 14].

Recently, one-dimensional quantum spin systems with next-nearest-neighbour interactions, which have been shown to result from the non-adiabaticity of the lattice distortion or the itineracy of electrons [15], have attracted much attention as a model of statistical mechanics

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and the many-body problem, since they include some phenomena of current interest. A typical example is the antiferromagnetic Heisenberg model with this kind of interaction, which gives rise to a degenerate ground state and the gapless-to-finite-gap phase transition [16], and the next-nearest-neighbour coupling is dependent on the critical exponents [17]. Subsequently, Nakano and Takahashi introduced long-range interaction to the quantum spin chain and obtained dimer states composed of nearest-neighbour singlet pairs as eigenstates of the system [18]. Our interest is in what the effect of this kind of interaction on the thermodynamic properties of the system is, and how the above results depend on the peculiar form of the next-nearest-neighbour interaction. On the other hand, it is also interesting to obtain an integrable two-element spin model in order to study two-element systems.

Lately, a new method has been adopted to study the isotropic multi-impurity model with next-nearest-neighbour interaction by Eckle *et al* [19], whose Hamiltonian has been transformed by using the Jordan–Wigner transform in the absence of backscattering. The main idea that allows the construction of the integrable multi-impurity model is that the Yang–Baxter equation continues to be satisfied under an arbitrary local shift in the spectral parameter $R(\lambda) \rightarrow R(\lambda+c)$. We find that this method can be used to construct a new integrable model [20], which includes a peculiar form of next-nearest-neighbour interactions and can be interpreted as a Hamiltonian of the two-element spin-chain model based on the isotropic Heisenberg spin chain [1, 21] in the periodic boundary case in view of the translational symmetry, where the next-nearest-neighbour interactions are introduced by adding a local shift in the spectral parameters. Our result shows that the additional parameter must be a real number, which is different from that in the open boundary case [4], to get a Hermitian Hamiltonian when the next-nearest-neighbour interactions exist and the magnetization in the z-direction and the total entropy of the system are independent of the additional next-nearest-neighbour interaction, but an additional energy is produced due to the local parameter shifts in the high-temperature limiting case. It is not difficult to generalize our model to the open boundary isotropic cases with multi-impurity [25] and next-nearest-neighbour interaction.

The present letter organized as follows. Our approach to the integrable model with next-nearest-neighbour interaction is developed in the next section. Our derivation of integral equations and the solutions are given in the section after that. Finally, some conclusions are drawn in the last section.

The model and the eigenvalue of the Hamiltonian

Now we devote our attention to constructing the model with the use of the periodic boundary condition. The monodromy matrix can be defined as

$$\Gamma_a(\lambda) = R_{a,1}(\lambda)R_{a,2}(\lambda + c/2)\cdots R_{a,2n-1}(\lambda)R_{a,2n}(\lambda + c/2)\cdots R_{a,2N-1}(\lambda)R_{a,2N}(\lambda + c/2) \tag{1}$$

for the periodic Heisenberg model [19], where we have taken the local shift as $c/2$ to simplify the calculation of the energy of the system. If we take some different local shift in the spectral parameter for some local vertices, we will obtain a model including multi-impurities. But we do not discuss that case here, since some results have been obtained by Eckle *et al* for multi-impurity cases. It is obvious that the equation satisfies the relation

$$R_{12}(\lambda_1 - \lambda_2)\overset{1}{\Gamma}(\lambda_1)\overset{2}{\Gamma}(\lambda_2) = \overset{2}{\Gamma}(\lambda_2)\overset{1}{\Gamma}(\lambda_1)R_{12}(\lambda_1 - \lambda_2)$$

and hence the transfer matrices $\tau(\lambda)$ are given by

$$\tau(\lambda) = \mathrm{tr}_a\, \Gamma_a(\lambda).$$

Using the relations (1) and

$$\tau^{-1}(0) = R^{-1}_{2n,2n+1}\left(\frac{c}{2}\right)\left\{\mathrm{tr}_a\, R_{a,1}(0)\cdots R_{a,2n-1}(0)R_{a,2n+1}(0)\cdots R_{a,2N-1}(0)R_{a,2N}\left(\frac{c}{2}\right)\right\}^{-1}$$
$$= R^{-1}_{2n,2n+1}\left(\frac{c}{2}\right)R^{-1}_{2n-1,2n+1}(0)\left\{\mathrm{tr}_a\, R_{a,1}(0)\cdots R_{a,2n-2}\right.$$
$$\left.\times\left(\frac{c}{2}\right)R_{a,2n+1}(0)\cdots R_{a,2N}\left(\frac{c}{2}\right)\right\}^{-1}$$

one obtains

$$Y = \tau^{-1}(\lambda)\,\frac{\partial}{\partial\lambda}\tau(\lambda)\bigg|_{\lambda=0} = \sum_{n=1}^{N}(y_{2n}+y_{2n-1}).$$

Let us keep in mind the periodic boundary conditions $\boldsymbol{\sigma}_{2N+1} = \boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_{2N+2} = \boldsymbol{\sigma}_2$; then we have

$$y_{2n} = R^{-1}_{2n,2n+1}(c/2)R'_{2n,2n+1}(c/2) = \frac{\mathrm{i}}{2(1+c^2/4)}\sum_{j=1}^{3}\sigma^j_{2n}\sigma^j_{2n+1} - \frac{\mathrm{i}}{2(1+c^2/4)}$$
$$y_{2n-1} = R^{-1}_{2n,2n+1}(c/2)R^{-1}_{2n-1,2n+1}(0)R'_{2n-1,2n+1}(0)R_{2n,2n+1}(c/2)$$
$$= \frac{\mathrm{i}}{2(1+c^2/4)}\sum_{j=1}^{3}\sigma^j_{2n-1}\sigma^j_{2n} + \frac{\mathrm{i}c^2}{2(4+c^2)}\sum_{j=1}^{3}\sigma^j_{2n-1}\sigma^j_{2n+1} - \frac{\mathrm{i}}{2} \tag{2}$$
$$+\frac{2c}{4+c^2}[\sigma^1_{2n-1}(\sigma^3_{2n}\sigma^2_{2n+1} - \sigma^2_{2n}\sigma^3_{2n+1}) + \sigma^2_{2n-1}(\sigma^1_{2n}\sigma^3_{2n+1} - \sigma^3_{2n}\sigma^1_{2n+1})$$
$$+\sigma^3_{2n-1}(\sigma^2_{2n}\sigma^1_{2n+1} - \sigma^1_{2n}\sigma^2_{2n+1})].$$

So the Hamiltonian of the system with periodic boundary conditions can be written as

$$H = -\frac{\mathrm{i}}{2}JY$$
$$H = \frac{1}{4}\sum_{n=1}^{2N}\frac{J}{1+c^2/4}(\sigma^1_n\sigma^1_{n+1} + \sigma^2_n\sigma^2_{n+1} + \sigma^3_n\sigma^3_{n+1})$$
$$+\frac{1}{4}\sum_{n=1}^{N}\frac{Jc^2}{4+c^2}(\sigma^1_{2n-1}\sigma^1_{2n+1} + \sigma^2_{2n-1}\sigma^2_{2n+1} + \sigma^3_{2n-1}\sigma^3_{2n+1}) \tag{3}$$
$$+\frac{1}{2}\frac{Jc}{4+c^2}[\sigma^1_{2n-1}(\sigma^3_{2n}\sigma^2_{2n+1} - \sigma^2_{2n}\sigma^3_{2n+1}) + \sigma^2_{2n-1}(\sigma^1_{2n}\sigma^3_{2n+1} - \sigma^3_{2n}\sigma^1_{2n+1})$$
$$+\sigma^3_{2n-1}(\sigma^2_{2n}\sigma^1_{2n+1} - \sigma^1_{2n}\sigma^2_{2n+1})] + D$$

where

$$D = -\frac{J}{4}\left(N + \sum_{n=1}^{N}\frac{1}{1+c_n^2}\right).$$

$\sigma^i_{2N+1} = \sigma^i_1$, and c must be real to keep the Hamiltonian Hermitian. The reason that we retain the coefficient $1/(1+c^2/4)$ in the Hamiltonian is to have the same form as in reference [22] for the energy when we take $c = 0$; this will make it easy to discuss the effect of the additional interaction. The terms in the first sum of equation (3) represent the next-neighbour exchange interaction among different sites and the others stand for the next-nearest-neighbour interaction. The above Hamiltonian describes an isotropic Heisenberg model with next-nearest-neighbour exchange interaction and can be explained as a model describing the many-body system with two kinds of particle, because one can see that (we take $\boldsymbol{\sigma}_{2kN+j} = \boldsymbol{\sigma}_j$)

$$H(\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_{2n}, \boldsymbol{\sigma}_{2n+1}, \ldots, \boldsymbol{\sigma}_{2N}) = H(\boldsymbol{\sigma}_{2j+1}, \ldots, \boldsymbol{\sigma}_{2n+2j}, \boldsymbol{\sigma}_{2n+2j+1}, \ldots, \boldsymbol{\sigma}_{2N+2j}) \qquad j = 1, 2, \ldots$$

but

$$H(\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_{2n}, \boldsymbol{\sigma}_{2n+1}, \ldots, \boldsymbol{\sigma}_{2N}) \neq H(\boldsymbol{\sigma}_{2j+2}, \ldots, \boldsymbol{\sigma}_{2n+2j+1}, \ldots, \boldsymbol{\sigma}_{2N+2j+1}) \qquad j = 1, 2, \ldots.$$

The above two equations mean that each unit cell should have two particles, which have different couplings with their next-nearest neighbours, in order to retain the translational periodicity of the Hamiltonian of the system. So the one-dimensional crystal lattice is not a simple one, as shown in figure 1, and its topological configuration is different from that of the simple periodic spin chain. This kind of configuration leads to exchange interaction of the A–B–A sites, but not of the B–A–B sites, so it describes a two-element spin system.

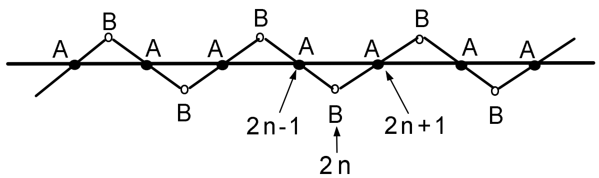

Figure 1. A pictorial representation of the interaction with two kinds of particle.

If $c \equiv 0$, the above Hamiltonian reduces to

$$H = \frac{J}{4} \sum_{n=1}^{2N} (\sigma_n^1 \sigma_{n+1}^1 + \sigma_n^2 \sigma_{n+1}^2 + \sigma_n^3 \sigma_{n+1}^3 - 1).$$

This is just the Hamiltonian of the isotropic Heisenberg model, which has been discussed by many authors (see, for example, [23,24]). We note that the Hamiltonian given by equation (3) represents integrable isotropic ferromagnetic (or antiferromagnetic) models with next-nearest-neighbour exchange interaction. To obtain an obvious result about the effect of this kind of interaction on the mean energy of the system, we use the algebraic Bethe *ansatz* to solve the spectrum $\tau(\lambda)$ for the periodic boundary case in the following. We define

$$\Gamma_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{4}$$

We know that for distinct values of λ and u,

$$\begin{aligned} [A(\lambda), B(u)] &= \mathrm{i}(\lambda - u)[B(u)A(\lambda) - B(\lambda)A(u)] \\ [D(\lambda), B(u)] &= \mathrm{i}(u - \lambda)[B(u)D(\lambda) - B(\lambda)D(u)] \end{aligned} \tag{5}$$

and as a result the $B(u)$ operators will serve as creation operators for eigenstates of $\tau(\lambda) = A(\lambda) + D(\lambda)$. Introducing a convenient basis:

$$|\Phi_{2N}\rangle = |e_1\rangle \otimes \cdots \otimes |e_{2N}\rangle \qquad |e_n\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_n \tag{6}$$

one has

$$\begin{aligned} &A(\lambda)\,|\Phi_{2N}\rangle = |\Phi_{2N}\rangle \qquad C(\lambda)\,|\Phi_{2N}\rangle = 0 \\ &D(\lambda)\,|\Phi_{2N}\rangle = \frac{[\mathrm{i}\lambda(\mathrm{i}\lambda + \mathrm{i}c/2)]^N}{[(\mathrm{i}\lambda + 1)(\mathrm{i}\lambda + \mathrm{i}c/2 + 1)]^N}\,|\Phi_{2N}\rangle\,. \end{aligned} \tag{7}$$

Define now

$$|\Psi(\lambda_1, \ldots, \lambda_M)\rangle = \prod_{l=1}^{M} B(\lambda_l)\, |\Phi_{2N}\rangle\,. \tag{8}$$

In the basis chosen, $|\Psi(\lambda_1, \ldots, \lambda_M)\rangle$ becomes an eigenstate:

$$\tau(\lambda)\, |\Psi(\lambda_1, \ldots, \lambda_M)\rangle = \Lambda(\lambda_1, \ldots, \lambda_M)\, |\Psi(\lambda_1, \ldots, \lambda_M)\rangle \tag{9}$$

if the parameters $\lambda_1, \ldots, \lambda_M$ satisfy the Bethe *ansatz* [22]

$$e^N(\Lambda_j) e^N(\Lambda_j + c) = \prod_{i=1\,(i\neq j)}^{M} e\left(\frac{\Lambda_j - \Lambda_i}{2}\right) \qquad j = 1, 2, \ldots, M \tag{10}$$

where $e(x) = (x + \mathrm{i})/(x - \mathrm{i})$ and $\Lambda_j = 2\lambda_j$. The corresponding eigenvalue is

$$\Lambda(\lambda_1, \ldots, \lambda_M) = \prod_{j=1}^{M} \frac{\mathrm{i}(\lambda_j - \lambda) + 1}{\mathrm{i}(\lambda_j - \lambda)} + \prod_{j=1}^{M} \frac{\mathrm{i}(\lambda - \lambda_j) + 1}{\mathrm{i}(\lambda - \lambda_j)} \frac{[\mathrm{i}\lambda(\mathrm{i}\lambda + \mathrm{i}c/2)]^N}{[(\mathrm{i}\lambda + 1)(\mathrm{i}\lambda + \mathrm{i}c/2 + 1)]^N}$$

and hence, finally, the energy eigenvalue is determined as [22]

$$E = -\sum_{j=1}^{M} \frac{2J}{\Lambda_j^2 + 1}. \tag{11}$$

Having found the eigenvalue, we try to formulate the equilibrium thermodynamics of the model in the following sections.

The thermodynamic properties in the limiting cases of zero and infinite temperature

In this section, we study the thermodynamic properties of the model [26,27]. Using the method of Takahashi [22], whose notation we will follow, we calculate the energy and magnetization in the z-direction per site. In the following, we will consider a system which is in an external magnetic field. The calculation is simple, and we only give the results since we have obtained the energy eigenvalues for the cases of no external field.

The energy and magnetization per site are

$$e = \frac{E}{2N} = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g_n(k)\rho_n(k)\, \mathrm{d}k - \mu_0 H \qquad g_n(k) = -\frac{2nJ}{k^2 + n^2} + 2n\mu_0 H \tag{12}$$

$$\frac{S_z}{2N} = \frac{1}{2} - \sum_{n=1}^{\infty} n \int_{-\infty}^{\infty} \rho_n(k)\, \mathrm{d}k \tag{13}$$

where H is magnetic field and the functions $\rho_n(k)$ satisfy the equations

$$\frac{1}{2\pi}\left(\frac{n}{(k+c)^2 + n^2} + \frac{n}{k^2 + n^2}\right) = \eta_n(k)\rho_n(k) + \sum_{m=1}^{\infty} A_{nm}\rho_m(k) \qquad (n = 1, 2, \ldots). \tag{14}$$

If $c \equiv 0$, the above equations reduce to those given by Takahashi [22]. Equation (14) can be transformed into

$$\frac{1}{2\pi}\left(\frac{1}{(k+c)^2 + 1} + \frac{1}{k^2 + 1}\right) = \eta_1(k)\rho_1(k) + \sum_{m=1}^{\infty} A_{1m}\rho_m(k) \tag{15}$$

$$\frac{1}{2\pi}\left(\frac{1}{(k+c)^2 + 1} + \frac{1}{k^2 + 1}\right) = ([0] + [2])(\eta_1(k) + 1)\rho_1(k) - [1]\eta_2(k)\rho_2(k) \tag{16}$$

$$[1](\eta_{n+2}(k)\rho_{n+2}(k) + \eta_n(k)\rho_n(k)) = ([0] + [2])(\eta_{n+1}(k) + 1)\rho_{n+1}(k) \tag{17}$$

and the functions $\eta_n(k)$ satisfy a set of coupled integral equations:

$$\begin{aligned}
&\ln(1+\eta_1(k)) = \frac{g_1(k)}{T} + \sum_{m=1}^{\infty} A_{1m}\ln(1+\eta_m^{-1}(k)) \\
&([0]+[2])\ln\eta_1(k) = -\frac{2J}{T}\frac{1}{k^2+1} + [1]\ln(1+\eta_2(k)) \\
&([0]+[2])\ln\eta_{n+1}(k) = [1]\{\ln(1+\eta_n(k)) + \ln(1+\eta_{n+2}(k)) \qquad (n=1,2,\ldots).
\end{aligned} \tag{18}$$

These are coupled integral equations which contain an infinite number of unknown functions, and it is not easy to solve them [22]. In the following we will solve them for some special cases.

We take the limit $J/T \to 0$ with the ratio H/T kept finite at first. In this case, it is evident that the $\eta_n(k)$ are all constants [22]; thus one has

$$\eta_n = \begin{cases} \left[\dfrac{z^{n+1}-z^{-n-1}}{z-z^{-1}}\right]^2 - 1 & \text{for } H/T > 0 \\ (n+1)^2 - 1 & \text{for } H/T = 0 \end{cases} \tag{19}$$

where $z = \exp(-\mu_0 H/T)$. Next we solve equations (15)–(17). Their Fourier transformations give that

$$\begin{aligned}
&(\mathrm{e}^{|\omega|} + \mathrm{e}^{-|\omega|}) f(n)\tilde{\rho}_n(\omega) = f(n-2)\tilde{\rho}_{n-1}(\omega) + f(n+2)\tilde{\rho}_{n+1}(\omega) \\
&\frac{1}{2}(\mathrm{e}^{-\mathrm{i}\omega c}+1)\mathrm{e}^{-|\omega|} = f(0)f(2)\tilde{\rho}_1(\omega) + (1+\mathrm{e}^{-2|\omega|})\sum_{n=1}^{\infty}\mathrm{e}^{-(n-1)|\omega|}\tilde{\rho}_n(\omega) \\
&-\frac{1}{2}(\mathrm{e}^{-\mathrm{i}\omega c}+1) = f(1)f(3)\tilde{\rho}_2(\omega) - f^{\,2}(1)\tilde{\rho}_1(\omega)(\mathrm{e}^{|\omega|}+\mathrm{e}^{-|\omega|})
\end{aligned} \tag{20}$$

where $f(n) = (z^{n+1} - z^{-n-1})/(z - z^{-1})$. Then we have that

$$\tilde{\rho}_n(\omega) = \frac{1}{2f(1)}\left\{\frac{\mathrm{e}^{-n|\omega|}}{f(n-1)f(n)} - \frac{\mathrm{e}^{-(n+2)|\omega|}}{f(n)f(n+1)}\right\}(\mathrm{e}^{-\mathrm{i}\omega c}+1) \tag{21}$$

or, in the k-representation,

$$\begin{aligned}
\rho_n(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega k}\tilde{\rho}_n(\omega)\,\mathrm{d}\omega = \rho_n^{(0)}(k) + \rho_n^{(c)}(k) \\
\rho_n^{(0)}(k) &= \frac{1}{\pi f(1)f(n)}\left\{\frac{1}{f(n-1)}\frac{n}{n^2+k^2} - \frac{1}{f(n+1)}\frac{n+2}{(n+2)^2+k^2}\right\} \\
\rho_n^{(c)}(k) &= \frac{1}{2\pi f(1)f(n)}\left\{\frac{1}{f(n-1)}\frac{n}{n^2+(k+c)^2} - \frac{1}{f(n+1)}\frac{n+2}{(n+2)^2+(k+c)^2}\right\} \\
&\quad - \frac{1}{2\pi f(1)f(n)}\left\{\frac{1}{f(n-1)}\frac{n}{n^2+k^2} - \frac{1}{f(n+1)}\frac{n+2}{(n+2)^2+k^2}\right\}.
\end{aligned} \tag{22}$$

We note that

$$\int_{-\infty}^{\infty}\rho_n^{(c)}(k)\,\mathrm{d}k = 0$$

so we have

$$\frac{S_z(c)}{2N} = \frac{S_z(0)}{2N} = \frac{1}{2} - \frac{z}{f(1)} = \frac{1}{2}\tanh\frac{\mu_0 H}{T}. \tag{23}$$

From reference [22], one has the expression for the entropy

$$\frac{S(c)}{2N} = \frac{S(0)}{2N} = \frac{1}{1+z^2}\ln(1+z^2) + \frac{1}{1+z^{-2}}\ln(1+z^{-2}). \tag{24}$$

The expressions for the magnetization and entropy are the same as those for $c = 0$—that is, the magnetization $S_z(c)$ and entropy $S(c)$ are independent of the next-nearest-neighbour interactions. This may be a result of the special form which results from the condition of integrability of the Hamiltonian given by equation (3), and the equation

$$\int_{-\infty}^{\infty} g(k+c)\,\mathrm{d}k = \int_{-\infty}^{\infty} g(k)\,\mathrm{d}k.$$

From (22), we get the energy per site

$$\begin{aligned} e(c) &= E/2N = e(0) + \Delta e(c) \\ e(0) &= \frac{J}{4}\left(\tanh^2\frac{\mu_0 H}{T} - 1\right) - \mu_0 H \tanh\frac{\mu_0 H}{T} \\ \Delta e(c) &= -\sum_{n=1}^{\infty} \frac{J}{f(1)f(n)}\left\{\frac{1}{f(n-1)}\left[\frac{2n}{4n^2+c^2} - \frac{1}{2n}\right] - \frac{1}{f(n+1)}\right. \\ &\quad \left. \times\left[\frac{(n+2)(4n+4+c^2)}{(2nc)^2+(4n+4+c^2)^2} - \frac{n(4n+4-c^2)}{(2nc)^2+(4n+4-c^2)^2} - \frac{1}{2(n+1)}\right]\right\}. \end{aligned} \tag{25}$$

Here we have used

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{n}{k^2+n^2}\frac{2n}{(k+c)^2+n^2}\,\mathrm{d}k = \frac{4n}{4n^2+c^2}$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{n}{k^2+n^2}\frac{2(n+2)}{(k+c)^2+(n+2)^2}\,\mathrm{d}k = \frac{2(n+2)(4n+4+c^2)}{(2nc)^2+(4n+4+c^2)^2} - \frac{2n(4n+4-c^2)}{(2nc)^2+(4n+4-c^2)^2}.$$

The numerical results for the additional energy $\Delta e(c)$ per site are given in figure 2.

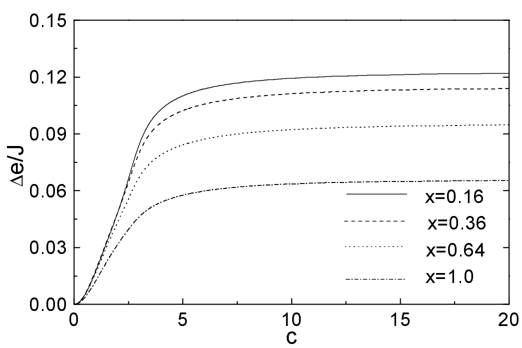

Figure 2. The additional energy $\Delta e(c)$ with varying coupling constant c.

We see in figure 2 that the additional energy $\Delta e(0) = 0$; the model reduces to the isotropic Heisenberg model with identical particles and has been discussed by many authors. For fixed x ($x = \mu_0 H/T$), $\Delta e(c)$ increases obviously with the increase of the coupling constant c when c is small ($c \lesssim 5$). If $c \gtrsim 5$, $\Delta e(c)$ increases slowly, and we note that

$$\Delta e(\infty) = -\frac{J}{8}\left(\tanh^2\frac{\mu_0 H}{T} - 1\right)$$

from equation (25). Then we have that

$$e(\infty) = \frac{J}{8}\left(\tanh^2\frac{\mu_0 H}{T} - 1\right) - \mu_0 H \tanh\frac{\mu_0 H}{T}.$$

It is easy to see this from equation (3), because one has

$$H \stackrel{c\to\infty}{\longrightarrow} \frac{J}{4}\sum_{n=1}^{N} J(\sigma_{2n-1}^1\sigma_{2n+1}^1 + \sigma_{2n-1}^2\sigma_{2n+1}^2 + \sigma_{2n-1}^3\sigma_{2n+1}^3 - 1).$$

This is just the isotropic Heisenberg model, but the interaction is only among the next-nearest neighbours and the total number of particles in the system is $2N$. It is also shown in figure 2 that for fixed c, $\Delta e(c)$ is dependent on the ratio $\mu_0 H/T$ and tends to $\Delta e(c)|_{x=0}$, where $\Delta e(c)|_{x=0}$ is given by equation (25) but taking $f(n) = n + 1$. So we have

$$0 \leqslant \Delta e(c) \leqslant \Delta e(c)\big|_{x=0} \leqslant J/8.$$

If $c \ll 1$, one has $\Delta e(c) = Kc^2 + \mathrm{O}(c^4)$, where $K = K(H/T)$ is a coefficient that is independent of c.

Now we consider the case where $T \to 0$. By following the method of [22], it is easy to see that if $J < \mu_0 H$, in the limit $T \to 0$ one has

$$\eta_n = \infty \qquad \rho_n = 0 \qquad \frac{S}{2N} = \frac{1}{2} \qquad \frac{E}{2N} = -\mu_0 H. \tag{26}$$

This solution means that all of the spins are parallel to the z-direction just like when there is no next-nearest-neighbour interaction. If $J > \mu_0 H$, in the limit $T \to 0$ one has

$$\begin{aligned}&\frac{1}{2\pi}\left[\frac{1}{(k+c)^2+1} + \frac{1}{k^2+1}\right] = \rho_1(k) + \frac{1}{\pi}\int_{-B}^{B} \frac{2\rho_1(k)}{4+(k-k')^2}\,\mathrm{d}k' \\ &\rho_n(k) = 0 \qquad n = 2, 3, \ldots\end{aligned} \tag{27}$$

where the parameter B is determined by $\varepsilon_1(B) = 0$, and $\varepsilon_1(k)$ is given by [22]

$$\varepsilon_1(k) = -\frac{\pi J}{2}\,\mathrm{sech}\,\frac{\pi k}{2} + \mu_0 H + \int_{\varepsilon_1(k')>0} R(k-k')\varepsilon_1(k')\,\mathrm{d}k'$$

where

$$R(x) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{1}{1+(x-y)^2}\,\mathrm{sech}\,\frac{\pi y}{2}\,\mathrm{d}y.$$

Then we have

$$\frac{S_z}{2N} = \frac{1}{2} - \int_{-B}^{B} \rho_1(k)\,\mathrm{d}k \tag{28}$$

$$\frac{E}{2N} = \int_{-B}^{B}\left(2\mu_0 H - \frac{2J}{k^2+1}\right)\rho_1(k)\,\mathrm{d}k - \mu_0 H. \tag{29}$$

If $c = 0$, equation (27) reduces to

$$\frac{1}{\pi}\frac{1}{k^2+1} = \rho_1(k) + \frac{1}{\pi}\int_{-B}^{B} \frac{2\rho_1(k)}{4+(k-k')^2}\,\mathrm{d}k'$$

which has been investigated by Griffiths [23] and by Yang and Yang [2]. Using their methods, one can calculate the magnetization curve and susceptibility of the one-dimensional Heisenberg spin chain and consider the effect of next-nearest-neighbour interaction on the magnetization under the condition of the low-temperature limit.

Conclusions

The Hamiltonian of a periodic boundary two-element spin system is derived on the basis of the isotropic Heisenberg spin model with next-nearest-neighbour interactions, and the energy, entropy, and magnetization of the system are obtained for some limits of the temperature and coupling constants. We find that the entropy and magnetization of the system are independent of the next-nearest-neighbour interaction in the high-temperature limit, but the energy increases with the increase of the coupling. The integral equations for the energy are also obtained in the zero-temperature-limit case. The method used in the present letter can be applied to the open boundary cases and the effects of the next-nearest-neighbour interaction on the energy and magnetization of the system in the low-temperature limit.

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